

QUASI-ISOMETRIES, BOUNDARIES AND JSJ-DECOMPOSITIONS OF RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. We demonstrate the quasi-isometry invariance of two important geometric structures for relatively hyperbolic groups: the coned space and the cusped space. As an application, we produce a JSJ-decomposition for relatively hyperbolic groups which is invariant under quasi-isometries and outer automorphisms.

1. INTRODUCTION

In [Gro87], Gromov introduced relatively hyperbolic groups as a generalization of δ -hyperbolic groups. The first author to elaborate on this idea was Farb, with a new definition in his thesis [Far94]. The equivalence between Gromov's definition and Farb's *relatively hyperbolic with bounded coset penetration* is established in [Bow97], and the necessity of BCP is demonstrated in [Szc98]. Following this, there have been contributions by Bowditch [Bow97] and many others, often describing these groups through new definitions [DS05, GM08, Osi06, Sis12, Yam04]. It is important to note that the notion of relative hyperbolicity only makes sense given a chosen collection \mathcal{A} of subgroups so it is more accurate to say that Γ is hyperbolic relative to \mathcal{A} . For instance, hyperbolic groups, which are naturally hyperbolic relative to $\{1\}$, can be investigated through their non-trivial relatively hyperbolic structures, e.g. the free group $F(a, b)$ is hyperbolic relative to $\langle [a, b] \rangle$.

These groups have enough structure to make them fruitful objects of study and they naturally occur in many contexts throughout modern geometric group theory. For instance:

- $\pi_1(M)$ for M a complete, finite volume manifold with pinched negative sectional curvature is hyperbolic relative to cusp subgroups [Far98, Bow98b];
- the fundamental group of a graph of groups with finite edge groups is hyperbolic relative to vertex groups [Bow98b];
- a limit group is hyperbolic relative to maximal non-cyclic abelian subgroups [Ali05, Dah03];
- a group acting geometrically on CAT(0) spaces with isolated flats is hyperbolic relative to the stabilizers of maximal flats [DS05, HK05];
- a hyperbolic group is hyperbolic relative to $\{1\}$;

For our purposes, the Groves-Manning description of relative hyperbolicity is ideal because of the combinatorial simplicity of computing path lengths. This computation, combined with the geometric insight provided by [Dru09], is the crucial component for the proof of our main result.

Theorem 4.3. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose that Γ_1 is hyperbolic relative to a finite collection \mathcal{A}_1 such that that no $A \in \mathcal{A}$ is properly relatively hyperbolic. Let $q : \Gamma_1 \rightarrow \Gamma_2$ be a quasi-isometry of groups. Then there exists \mathcal{A}_2 , a collection of subgroups of Γ_2 , such that the cusped space of $(\Gamma_1, \mathcal{A}_1)$ is quasi-isometric to that of $(\Gamma_2, \mathcal{A}_2)$.*

We remark that there is an analogue for the coned space (Theorem 4.4). Several corollaries follow from Theorem 4.3, including Corollary 4.7: exchanging finite generating sets does not change the quasi-isometry type of the cusped space, (Definition 2.3). However, our main application of Theorem 4.3 stems from the following:

Corollary 4.5. With $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ as in Theorem 4.3, the cusped spaces $X(\Gamma_1, \mathcal{A}_1)$ and $X(\Gamma_2, \mathcal{A}_2)$ have homeomorphic boundaries.

In [Bow98a, Bow01], the actions of hyperbolic and relatively hyperbolic groups on their boundaries are analyzed to produce JSJ and peripheral splittings, respectively. The topological features of the boundary (cut pairs and cut points, Definition 3.2) correspond to the limit sets of two-ended and peripheral subgroups in the boundary, respectively.

This approach was generalized to include groups acting on continua in [PS06]. The action is inherited by an \mathbb{R} -tree which is constructed from the topological features of the continuum (again, cut pairs and points). We pause to note that even in the context of hyperbolic groups this strategy cannot be extended to produce different splittings from larger separating sets such as Cantor sets, [DP10]. An application for this construction to CAT(0) groups appears in [PS09]. Here, the group action on the CAT(0) boundary is analyzed via a variant of the convergence property to determine the isomorphism types of the vertex groups.

In [PS06], the authors suggest that the cut-point/cut-pair tree may be useful in the analysis of relatively hyperbolic groups. We demonstrate that this is true. On our way to this, we prove that this tree is simplicial in Theorem 3.11. Further, the splitting produced satisfies our Definition 2.16 (for more generality, see [GL10a]) of a JSJ splitting over elementary subgroups relative to peripheral subgroups, Theorem 3.12.

We also note that in Section 1 of [Bow01] a project to extend the results of [Bow98a] to relatively hyperbolic groups is suggested. Here, the terminology *local cut points* is used in place of our *cut pairs*, Definition 3.2. By using the convergence property we are able to identify the vertex groups of the combined tree. Since this is a JSJ tree and it is quasi-isometry invariant, we effectively finish this project.

Theorem 5.5. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose additionally that Γ_1 is one-ended and hyperbolic relative to the finite collection \mathcal{A}_1 of subgroups such that no $A \in \mathcal{A}$ is properly relatively hyperbolic or contains an infinite torsion subgroup. Let \mathcal{T} be the cut-point/cut-pair tree of $\partial(\Gamma_1, \mathcal{A}_1)$. If $f : \Gamma_1 \rightarrow \Gamma_2$ is a quasi-isometry then*

- \mathcal{T} is the cut-point/cut-pair tree for Γ_2 with respect to the peripheral structure induced by Theorem 4.3,
- if $\text{Stab}_{\Gamma_1}(v)$ is one of the following types then $\text{Stab}_{\Gamma_2}(v)$ is of the same type,
 - (1) hyperbolic 2-ended,
 - (2) peripheral,
 - (3) relatively QH with finite fiber.

Moreover, by [MOY12, Corollary 7.3] the set of relatively hyperbolic structures on Γ forms a partial order and by our condition on the parabolic subgroups, this peripheral structure is the unique maximal structure. Combining this with Corollary 4.5, we see that the action of $\text{Out}(\Gamma)$ induces homeomorphisms of $\partial\Gamma$ and preserves the JSJ tree. Thus, we get the following corollary.

Corollary 5.6. The JSJ tree corresponding to the cut-point / cut-pair tree is a fixed point for the action of $\text{Out}(\Gamma)$ on the JSJ deformation space.

2. PRELIMINARIES

The following section is a somewhat terse summary of the various definitions and facts which we will use directly. We also include some discussion relating our results to other research. Given this brevity, we suggest several more comprehensive resources: for coarse geometry, δ -hyperbolic spaces and groups, and their boundaries, see [BH99]; for relatively hyperbolic groups, [Bow97, Dru09, Far98, GM08, Hru10, Osi06]; for constructing group actions on trees from group actions on continua [Bow98a, Bow98b, PS06]; for JSJ decompositions, [GL10a, GL10b]; and for convergence actions [GM87]. For the remainder of this paper, we assume that Γ is generated by the finite set S .

2.1. Relatively Hyperbolic Groups and $X(\Gamma, \mathcal{A})$.

Definition 2.1. [GM08] Let T be any graph with edges of length 1. We define the *combinatorial horoball* based at T , $\mathcal{H}(= \mathcal{H}(T))$ to be the following 1-complex:

- $\mathcal{H}^{(0)} = VT \times \{\{0\} \cup \mathbb{N}\}$
- $\mathcal{H}^{(1)} = \{((t, n), (t, n+1))\} \cup \{((t_1, n), (t_2, n)) \mid d_T(t_1, t_2) \leq 2^n\}$. We call edges of the first set *vertical* and of the second *horizontal*.

Remark. In [GM08], the combinatorial horoball is described as a 2-complex because they needed the complex to be simply connected. As we are not concerned with that here, we only require the 1-skeleton.

Definition 2.2. [GM08] Let $D : \mathcal{H} \rightarrow [0, \infty)$ be defined by extending the map on vertices $(t, n) \rightarrow n$ linearly across edges. We call D the *depth function* for \mathcal{H} and refer to vertices v with $D(v) = n$ as *vertices of depth n* or *depth n vertices*.

Because $T \times \{0\}$ is homeomorphic to T , we identify T with $D^{-1}(0)$.

Definition 2.3. [GM08] Let \mathcal{A} be a collection of subgroups of Γ . The *cusped space* $X(\Gamma, S, \mathcal{A})$ is the union of Γ with $\mathcal{H}(gA)$ for every coset of $A \in \mathcal{A}$, identifying gA with the depth 0 subset of $\mathcal{H}(gA)$. We suppress mention of S, \mathcal{A} when they are clear from the context.

For points in $X(\Gamma)$, we do not distinguish between the depth functions of distinct horoballs because horoballs only overlap at depth 0 and so this convention is unambiguous. Thus, we can refer to the depth of any vertex in $X(\Gamma)$ without mention of the associated horoball or coset.

Definition 2.4. The elements of the collection of subgroups \mathcal{A} are called *parabolic subgroups* and the subgroups which are conjugate to them are called *peripheral subgroups*.

Definition 2.5. [GM08] If $X(\Gamma, S, \mathcal{A})$ is δ -hyperbolic then we say that Γ is *hyperbolic relative to \mathcal{A}* or that Γ is a *relatively hyperbolic group* or that *the pair (Γ, \mathcal{A}) is relatively hyperbolic*.

Remark. As mentioned in the introduction, there are many definitions of relatively hyperbolic groups. These definitions are all equivalent, with the exception of Farb's in which the phrase "with bounded coset penetration" needs to be appended. Equivalence of Definition 2.3 with others is established in [GM08, Theorem 3.25].

Remark. Substituting S for some other finite generating set S' changes the topology of $X(\Gamma, S, \mathcal{A})$ and may change the value of δ , but does not affect the hyperbolicity of the cusped space for some δ' . Thus we omit S from the definition above. See [GM08], or [Osi06] for this result in a different context.

Remark. In [Hru10, Section 4], the construction of the cusped space is extended to groups which only have a finite *relative generating set* (ie $\Gamma = \langle S \cup H \rangle$) by declaring a proper metric on peripheral subgroups. It seems reasonable to generalize our results in this fashion.

Definition 2.6. [Bow97] Given a group Γ hyperbolic relative to \mathcal{A} , the *Bowditch boundary*, $\partial(\Gamma, S, \mathcal{A})$ is the Gromov boundary of the associated cusp space, $\partial X(\Gamma, S, \mathcal{A})$. When there is no ambiguity we simply say the boundary.

Remark. In [Bow97], the boundary is defined as the ideal boundary of a proper, hyperbolic space on which the group acts. This is largely motivated by the definition for relatively hyperbolic groups given in [Gro87]. Part of the appeal of the characterization of relative hyperbolicity given in [GM08] is that it satisfies the conditions of the definition of [Gro87], and so it naturally fits with the notion of boundary developed in [Bow97]. In fact, Bowditch constructs a very similar space to the cusped space in which the edges at depth n are shrunk by a factor of 2^n instead of adding extra edges. The two spaces are quasi-isometric by the natural map on vertices, so the boundaries are the same.

Lemma 2.7. [GM08, 3.11] If A is a combinatorial horoball, then the Gromov boundary consists of a single point, denoted e_A , which can be represented by a geodesic ray containing only vertical edges.

Because our proof of Theorem 4.3 can be simplified to prove an analogous result for the coned space, we include its definition.

Definition 2.8. [Bow97] Let G be a group with a finite collection of subgroups \mathcal{B} . For every coset gB_i of an element of \mathcal{B} , we add to the Cayley graph a vertex v_{gB_i} and from every element of gB_i we add an edge to v_{gB_i} . We call this the *coned space* of (G, \mathcal{B}) .

Definition 2.9. [Bow97] A graph is *fine* if for every $n > 0$ and every edge e , the number of cycles of length at most n containing e is finite.

Definition 2.10 ([Bow97] Alternate characterization of relative hyperbolicity). Let C be the Cayley graph of a group G generated by a finite set S and let \mathcal{A} be a collection of subgroups of G . If the coned space of (G, \mathcal{A}) is hyperbolic and fine, then (G, \mathcal{A}) is relatively hyperbolic [Bow97, p. 27].

The following three definitions appear in Chapter 4 of [Osi06].

Definition 2.11. An element g of is called *hyperbolic* if g is not conjugate into any $A_i \in \mathcal{A}$.

Definition 2.12. A subgroup H of a relatively hyperbolic group $\Gamma = \langle S | R \rangle$ is called *relatively quasi-convex* if there exists $\sigma > 0$ such that the following condition holds. Let f, g be two elements of H , and p an arbitrary geodesic path from f to g in $\text{Cay}(\Gamma, S \cup \mathcal{A})$. Then for any vertex $v \in p$, there exists a vertex $w \in H$ such that $\text{dist}_S(u, w) \leq \sigma$.

Definition 2.13. If H is relatively quasi-convex and $H \cap A^g$ is finite for all $g \in \Gamma$ and $A \in \mathcal{A}$, then H is called *strongly relatively quasi-convex*.

Theorem 2.14 ([Osi06], Theorem 4.19). *The centralizer of a hyperbolic element in a relatively hyperbolic group is strongly relatively quasi-convex.*

Definition 2.15. A subgroup of a relatively hyperbolic group is called *elementary* if it is either virtually cyclic or a subgroup of a peripheral subgroup.

2.2. JSJ Decompositions. JSJ decompositions were first introduced in the context of 3-manifold topology in the works of Jaco and Shalen and, independently, Johannson [JS79, Joh79]. The manifold is separated by a maximal system of disjoint, embedded tori with the goal of understanding the structure of the ambient manifold by inspecting the individual components.

This idea was ported to geometric group theory originally by Kropholler [Kro90] and then by Rips and Sela, [RS97]. Several authors have expanded on these ideas, including [Bow98a, DS99, FP06, Pap05, PS09], and, in a slightly different manner, [SS03]. In the group theoretic setting, JSJ decompositions encode a maximal amount of the information related to two-ended splittings (or, in the case of [RS97], \mathbb{Z} -splittings) as a graph of groups.

Several of these notions have recently been unified by the results in [GL07, GL10a, GL10b]. It is the language presented here which we adopt. Since we are interested only in a particular type of JSJ-decomposition, we do not include the most general definitions.

Definition 2.16. Let Γ be hyperbolic relative to \mathcal{A} . An *elementary JSJ splitting relative to \mathcal{A}* is a tree, T , with a Γ action such that the following hold.

- (1) all edge stabilizers are elementary subgroups;
- (2) (universally elliptic) any edge stabilizer of T fixes a point in any other tree with property (1);
- (3) (maximal for domination) for any tree T' satisfying (2), every vertex stabilizer of T stabilizes a vertex of T' ; and
- (4) (relative to \mathcal{A}) all subgroups of elements of \mathcal{A} fix a point in T .

In [GL10a], the authors promote the notion that the correct objects of study for JSJ decompositions should not be JSJ trees because they are not unique. Rather, a collection of trees is more fundamental:

Definition 2.17. A maximal collection of trees which all satisfy Definition 2.16 is called a *JSJ deformation space*.

In the study of JSJ decompositions, one important focus is on understanding those subgroups in which there are many mutually incompatible splittings. Such pairs of splittings are called *hyperbolic-hyperbolic* in [RS97] and are best understood as analogous to splittings of a surface group over two simple closed curves with an essential intersection. It is impossible to realize both splittings simultaneously with a common refinement of the graphs of groups.

The subgroups with this property have various names in different contexts, including *quadratically hanging* [RS97], *maximal hanging Fuchsian* [Bow98a], *orbifold hanging vertex* [Pap05]. These names all seek to describe the same central idea: many pairs of simple closed curves on surfaces intersect. Consequently, surface groups have many \mathbb{Z} -splittings which can not be simultaneously realized in a common graph of groups. The essential power of these ideas is that whenever these ‘hyperbolic-hyperbolic’ splittings occur in finitely presented groups, the situation is always very close to the surface case. There is a more general definition given in [GL10a] which happens to encompass all of the above.

Definition 2.18. Let Γ_v be a vertex group for a JSJ tree. If Γ_v is not universally elliptic, then Γ_v is called *flexible*.

For our purposes, this definition falls somewhat short. We are interested in identifying subgroups up to quasi-isometry and flexibility is not preserved under such maps. For example, compare closed hyperbolic surface groups with hyperbolic triangle groups. In the context of relatively hyperbolic groups, there is a larger class of subgroups which we can use instead of flexible subgroups and which will reflect the coarse geometry directly: relatively QH subgroups with finite fiber [GL10a].

Definition 2.19. Given a group with a JSJ tree relative to \mathcal{A} , a subgroup Q is a *relatively QH-subgroup* if it satisfies the following:

- (1) an exact sequence, with \mathcal{O} a hyperbolic 2-orbifold, F called the *fiber*:

$$1 \rightarrow F \rightarrow Q \rightarrow \pi_1(\mathcal{O}) \rightarrow 1$$

- (2) the images of incident edge groups are either finite or contained in a boundary subgroup of $\pi_1(\mathcal{O})$.
- (3) every conjugate of an element of \mathcal{A} intersects Q with image either finite or contained in a boundary component of $\pi_1(\mathcal{O})$.

Specifically, these subgroups will be detectable from the boundary of the group, and thus will be realized as vertex groups for the cut-point/cut-pair tree.

2.3. Convergence Actions. Suppose Γ acts on a space X by homeomorphisms.

Definition 2.20. A sequence (g_i) is a *convergence sequence on X* if there exists points $x_1, x_2 \in X$ such that for any compact C not containing x_1 , $g_i(C) \rightarrow x_2$.

Definition 2.21. If every sequence in G contains a convergence subsequence, then G acts as a convergence group on X .

Both hyperbolic and relatively hyperbolic groups are characterized by types of convergence actions which they exhibit on their boundaries [Bow98b, Yam04]. This allows for the identification of stabilizers of certain topological features in these contexts. We also require the classic theorem

Theorem 2.22 ([CJ94, Gab92, Tuk88]). *Let G be a subgroup of $\text{Homeo}(S^1)$. G acts as a convergence group on S^1 if and only if G is Fuchsian.*

A *Fuchsian group* is a discrete subgroup of Möbius transformations of D^2 . In the proof of Theorem 4.3 we will use the convergence action to identify the vertex groups of our splitting.

3. THE TREE FROM THE BOUNDARY

Given a group acting on a continuum by homeomorphisms, [PS06] constructs an \mathbb{R} -tree with vertices representing the topological features of the continuum (cut points and cut pairs). The tree inherits the action of the group on the continuum. We condense their exposition and demonstrate that this tree is simplicial and of JSJ type in the context of relative hyperbolicity. For the remainder of this section we assume that Γ is a relatively hyperbolic group.

Definition 3.1. A *continuum* is a compact connected metric space.

Definition 3.2. Given a continuum X , a point $x \in X$ is a *cut point* if $X \setminus \{x\}$ is not connected. If $\{a, b\} \subset X$ contains no cut points and $X \setminus \{a, b\}$ is not connected, then $\{a, b\}$ is a *cut pair*. A set Y is called *inseparable* if no two points of Y lie in different components of the complement of any cut pair.

These separating features occur as the fixed points of peripheral or hyperbolic two-ended subgroups over which Γ splits. Given that we want to also understand when there are many mutually incompatible splittings (as in Definitions 2.18 and 2.19), we have terminology reflecting interlocking cut pairs. These pairs arise in our context as the endpoints of pairs of hyperbolic two-ended subgroups over which the group splits but which admit no common refinement.

Definition 3.3. Let X be a continuum without cut points. A finite set S is called a *cyclic subset* if there is an ordering $S = \{s_1, s_2, \dots, s_n\}$ and continua M_1, \dots, M_n such that

- (1) $M_i \cap M_{i+1} = \{s_i\}$, subscripts mod n
- (2) $M_i \cap M_j = \emptyset$ whenever $|i - j| > 1$
- (3) $\cup M_i = X$

An infinite subset in which all finite subsets of cardinality at least 2 are cyclic is also called cyclic.

Definition 3.4. A maximal cyclic subset with at least 3 elements is called a *necklace*.

Cyclic subsets arise as collections of mutually separable cut pairs. We also note that an inseparable cut pair can be in the closure of more than one necklace, but if the cut pair is not inseparable then that necklace is unique.

Given a continuum X , we define an equivalence relation \sim such that any cut point is equivalent only to itself and for x, y which are not cut pairs, $x \sim y$ if and only if there is no cut point z such that x and y are in different components of $X \setminus z$.

We would like to define a similar notion for cut pairs but the extra structure makes this difficult. Instead, we directly construct subsets of the powerset of X which reflect the topology. Let \mathcal{R} be the collection containing all inseparable cut pairs, necklaces and maximal inseparable sets of X . We claim that this structure is compatible with \sim , ie that \mathcal{R} is the union of sets defined similarly on each class of \sim . This follows from the following lemma, that cut points do not separate cut pairs.

Lemma 3.5. Suppose that T is a connected topological space with cut point x . If $\{y, z\}$ is a cut pair then y and z are in the same component of $T \setminus x$.

Proof. Let C_1 be the component of $T \setminus \{y, z\}$ containing x . Let w be a point in another component, C_2 . Clearly x separates C_1 but not C_2 . Thus, w and y are in the same component of $T \setminus \{x, z\}$ and w and z are in the same component of $T \setminus \{x, y\}$. Thus, y and z are in the same component of $T \setminus x$. \square

In [PS06, Theorems 12, 13, 14], \sim is shown to satisfy a ‘betweenness’ property so that a process of ‘connecting the dots’ can fill it in to an \mathbb{R} -tree. [PS06, Corollary 31] serves the same purpose for \mathcal{R} . The combination of the two (which Lemma 3.5 justifies) is discussed in Section 5 of [PS06], see Figure 1. Obviously a group action by homeomorphisms on a continuum is inherited by this tree, although the action is not *a priori* isometric. In fact, this \mathbb{R} -tree does not necessarily come equipped with a metric.

Of course, the most important question about an action of a group on an \mathbb{R} -tree is whether it can be promoted to an action on a simplicial tree, or whether the \mathbb{R} -tree itself is simplicial. This

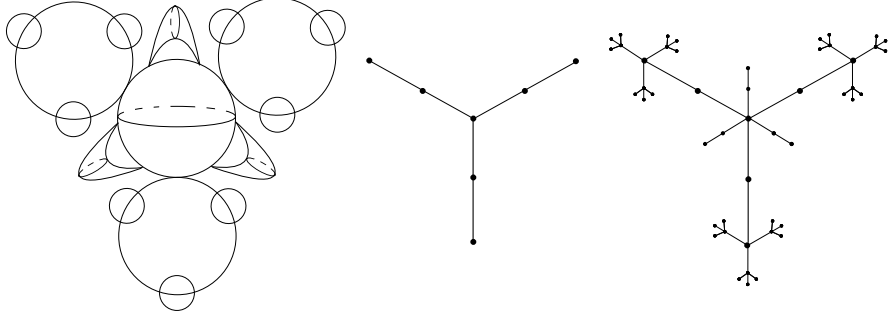


FIGURE 1. A continuum with the associated cut-point tree and combined tree.

is often accomplished via the Rips machine, which can be applied in the context of a reasonably nice action.

Definition 3.6. Let Γ act on the \mathbb{R} -tree T by homeomorphisms. We say the action is *nesting* if there exists a $g \in \Gamma$ and an interval $I \subset T$ such that $g(I)$ is properly contained in I . Otherwise, we say the action is *non-nesting*.

A non-degenerate arc I is called *stable* if there is a non-degenerate sub-arc such that for any non-degenerate arc $K \subset J$, $\text{Stab}(K) = \text{Stab}(J)$. An action is called *stable* if any closed arc I of T is stable.

In order to show that our action is stable, we require some information on the variety of finite subgroups of a relatively hyperbolic group.

Lemma 3.7. Let Γ be a relatively hyperbolic group. There are finitely many conjugacy classes of finite order subgroups F such that F is contained in a hyperbolic two-ended subgroup H with F fixing the ends of H .

Proof. The following argument is adapted from [Mos12]. For every hyperbolic two-ended subgroup H , take $A_H \subset X(\Gamma, \mathcal{A})$ to be the set of all geodesics between the endpoints of H . There is a uniform width W for all A_H which is independent of the choice of H [Hru10, Corollary 8.16], ie for a point $p \in A_H \cap \text{Cay}(\Gamma) \subset X(\Gamma, \mathcal{A})$, $H \setminus N_W(p)$ is not connected.

It follows that for some $k \in \mathbb{N}$, if $h \in H$ has the property that $[h.N_{kW}(p)] \cap N_{kW}(p) = \emptyset$ and h fixes the endpoints of H , then h has infinite order.

For F as above, it must be that the F -translates of $N_{kW}(p)$ are not disjoint from $N_{kW}(p)$. Conjugating by p sends every such F to a subgroup $F' \subset N_{kW}(1)$ of the same conjugacy class, and there are only finitely many possible F' . \square

We remark that the cut-point tree (ie the tree produced by connecting the dots for \sim) is simplicial whenever the boundary is connected and locally connected [Bow01, Theorem 9.2]. This can be achieved by the following mild constraints on \mathcal{A} .

Theorem 3.8 ([Bow01], Theorem 1.5). *Suppose that Γ is relatively hyperbolic and that each peripheral subgroup is one- or two-ended and contains no infinite torsion subgroup. If Γ is connected then it is locally connected.*

We can produce a tree by ‘blowing up’ the vertices of subcontinua with no cut points according to the cut-pair structure (\mathcal{R}) , as discussed in [PS06, Section 5]. We call this the *combined tree* or *cut-point / cut-pair tree*, which we will denote by \mathcal{T} (Figure 1). As mentioned above, we seek to apply the Rips machine so we demonstrate that the action is stable and non-nesting.

Lemma 3.9. The action of Γ on \mathcal{T} is stable.

Proof. Because the cut point tree is simplicial, the only intervals which can be unstable are those which contain multiple inseparable cut pairs. Let I be any interval containing at least two inseparable cut pairs, A, B . We show that $\text{Stab}(I)$ is finite. If $\{g_n\} \subset \text{Stab}(I)$ is an infinite

sequence of elements then we may assume that $g_n(x) \rightarrow p$ for all $x \in \partial\Gamma$, perhaps after passing to a subsequence. However, each g_n must fix all of the points of both cut pairs. This is a contradiction.

Additionally, this finite subgroup satisfies the hypotheses of Lemma 3.7 because it is a subgroup of $\text{Stab}(A)$. There are only finitely many conjugacy classes of such subgroups so there is a uniform bound on the order of $\text{Stab}(I)$. It follows immediately that the action is stable. \square

Lemma 3.10. The action of Γ on \mathcal{T} is non-nesting.

Proof. Assume not. Then there exists an interval $I \subset T$ and $g \in \Gamma$ (replaced by g^2 if necessary) such that $g(I)$ is a proper subset of I . By the Brouwer fixed point theorem, there is a fixed point of g in I , call this A . We may assume $I = [A, B]$, $g(B) \in [A, B]$ and that g has infinite order.

By convergence, there exists $p, q \in \partial\Gamma$ such that $g^n(x) \rightarrow p$ for every $x \neq q$. Clearly, $p \in A$. Because $g^{-n}(x) \rightarrow q$, also $q \in A$. However, this implies that for all $x \neq p$, $g^{-n}(x) \rightarrow q \in A$. Yet $g^{-n}(B) \notin [A, B]$ for any n , a contradiction. \square

Remark. Lemma 3.10 precludes the possibility that a cut point can be an end of the cut-pair tree. Topologically, this situation is exhibited in the following example provided by Eric Swenson, Figure 2. Let M be the closure of a union of infinitely many increasingly small ellipses centered at the origin which intersect in exactly two points. For instance, if the minor axis of one is the same length as the major axis of its successor and they are arranged at an angle of $\pi/2$. Let X be the union of two copies of M , connected at their center points by a thickened arc. The combined tree can never be simplicial because between any cut pair and a cut point there are infinitely many cut pairs. This pathology can never occur in the boundary of a relatively hyperbolic group because the action of the stabilizer of a cut point would necessarily be nesting. Thus, even if the cut point tree is simplicial and all cut pair trees are simplicial, we still require the convergence action to prove that the combined tree is simplicial.

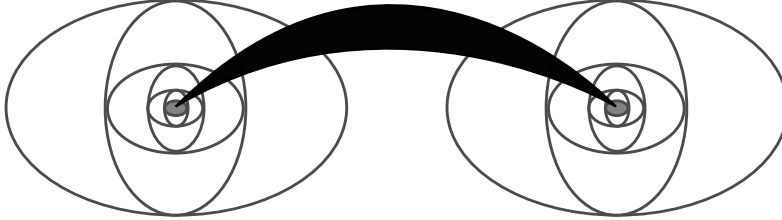


FIGURE 2. A space with simplicial cut point and cut pair trees but without a simplicial combined tree.

Theorem 3.11. Let Γ be a one-ended group hyperbolic relative to \mathcal{A} such that for every $A \in \mathcal{A}$, A is not properly relatively hyperbolic and A contains no infinite torsion subgroup. Let \mathcal{T} be the combined tree obtained by the action of Γ on its Bowditch boundary. Then \mathcal{T} is simplicial.

Proof. Suppose not. Then because the action is non-nesting, by [Lev98], there is an \mathbb{R} -tree \mathcal{T}' equipped with an isometric Γ -action and an equivariant quotient map $\mathcal{T} \rightarrow \mathcal{T}'$. Furthermore, stabilizers of segments in \mathcal{T}' stabilize segments in \mathcal{T} , and so are finite of uniformly bounded order. Therefore, as in Lemma 3.9, the Γ -action is stable.

In all cases of [BF95, Theorem 9.5] other than the pure surface case, one obtains a splitting over a finite group. However, Γ is one-ended, so we reduce to this case. By [BF95, Theorems 9.4(1) & 9.5] Γ admits a splitting over a two-ended group V , and this two-ended group corresponds to an essential, non-boundary parallel simple closed curve in the associated orbifold. If $g \in V$ corresponds to this curve, then since the associated lamination on the orbifold has no closed leaves g must act hyperbolically on \mathcal{T}' . This implies that g also acts hyperbolically on \mathcal{T} . However, a splitting of Γ over a two-ended group must induce a cut pair corresponding to the

endpoints of the axis of $\langle g \rangle$. This cut pair must be stabilized by g , so g cannot act hyperbolically. This is a contradiction. \square

In summary, the combined tree \mathcal{T} is simplicial and has one vertex for each of the following topological structures in the continuum:

- (1) cut points
- (2) inseparable cut pairs
- (3) necklaces
- (4) equivalence classes of points not separated by cut points or cut pairs

Additionally, there is an edge between two vertices if the corresponding sets in the continua have intersecting closures.

Remark. Because groups acting on CAT(0) spaces with isolated flats are hyperbolic relative to the stabilizers of maximal flats, our results capture a finer splitting than those presented in [PS09] when restricted to this setting. Every two-ended splitting is over either a hyperbolic or a peripheral subgroup so those splittings are always detectable in the boundary of the cusped space. Thus, our tree has a vertex corresponding to every vertex of their tree, as well as vertices for splittings over one-ended peripheral subgroups. However, we do lose the capability of recognizing two-ended splittings as such when they are realized as peripheral splittings, see Figure 3.

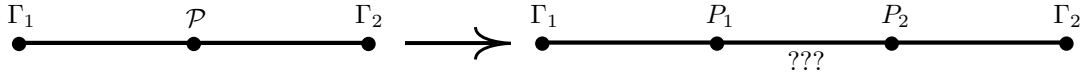


FIGURE 3. A peripheral splitting may mask a two-ended splitting.

Now that we know that the tree is simplicial, we can characterize it according to [GL10a].

Theorem 3.12. *The cut-pair/cut-point tree \mathcal{T} is a JSJ tree over elementary subgroups relative to peripheral subgroups.*

Proof. We show that \mathcal{T} satisfies the conditions of Definition 2.16. By the construction of the tree every edge group must be the stabilizer of either a cut point or a cut pair. Because relatively hyperbolic groups act on their boundaries with a convergence action, these stabilizers must be elementary subgroups (condition (1)). Every peripheral subgroup fixes a point in the tree because it fixes a point in the boundary (this point is e_A of Lemma 2.7), which implies condition (4). Furthermore, this tree satisfies (3) because every elementary splitting always has a topological expression in the boundary. In particular, [Bow01] implies the existence of a cut point and [Pap05] implies the existence of a cut pair whenever there is an peripheral or hyperbolic two-ended splitting, resp. Thus, every vertex in every such tree comes from one of these structures and hence is already a vertex stabilizer in \mathcal{T} . Finally, every splitting of this group must reflect the topology of the boundary. In particular, every edge represents the intersection of topological features. Fixing these features means fixing a vertex in a tree for another splitting. This is (2). \square

4. PROOF OF THE MAIN THEOREM

In this section we prove our main result:

Theorem 4.3. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose that Γ_1 is hyperbolic relative to a finite collection \mathcal{A}_1 such that that no $A \in \mathcal{A}$ is properly relatively hyperbolic. Let $q : \Gamma_1 \rightarrow \Gamma_2$ be a quasi-isometry of groups. Then there exists \mathcal{A}_2 , a collection of subgroups of Γ_2 , such that the cusped space of $(\Gamma_1, \mathcal{A}_1)$ is quasi-isometric to that of $(\Gamma_2, \mathcal{A}_2)$.*

We first show that horoballs of quasi-isometric spaces are themselves quasi-isometric. To that end, we distinguish among types of geodesics which exist in horoballs. We assume that $n_2 > n_1$.

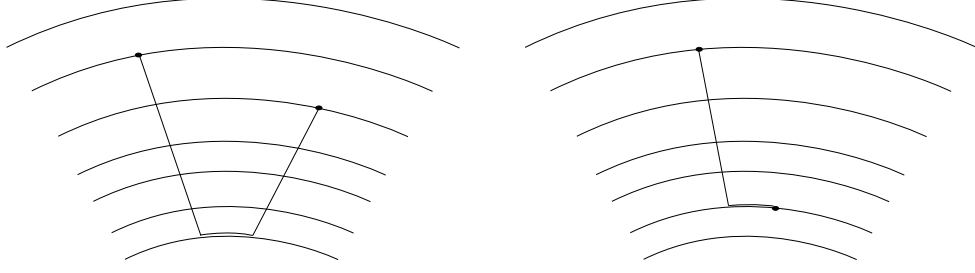


FIGURE 4. A vertical geodesic (right) and a non-vertical geodesic.

Definition 4.1. Let \hat{T} be a horoball over the graph T with (t_1, n_1) and (t_2, n_2) vertices of \hat{T} . We say $[(t_1, n_1), (t_2, n_2)]$ is *vertical* or a *vertical geodesic* if n_2 is the maximal depth among vertices of $[(t_1, n_1), (t_2, n_2)]$. See Figure 4.

Lemma 4.2. Let $q : T \rightarrow S$ a (k, c) -quasi-isometry between graphs. There is a $(1, C)$ -quasi-isometry $\hat{q} : \hat{T} \rightarrow \hat{S}$ between combinatorial horoballs such that \hat{q} extends q . Furthermore, C depends only on k and c .

Proof. Extend q to \hat{q} by defining $\hat{q}(v, n) = (q(v), n)$ and let $s_i = q(t_i)$. We partition the proof into cases by which geodesics are vertical.

$[(s_1, n_1), (s_2, n_2)]$ *vertical*: $d_{\hat{S}} \in \{n_2 - n_1 + 1, n_2 - n_1 + 2, n_2 - n_1 + 3\}$ and $d_{\hat{T}} + 3$ is clearly at least as large.

For the remaining two cases we assume that $[(s_1, n_1), (s_2, n_2)]$ is not vertical.

$[(t_1, n_1), (t_2, n_2)]$ *not vertical*:

$$\begin{aligned} d_{\hat{S}}((s_1, n_1), (s_2, n_2)) &\leq 2 \log_2 [d_S(s_1, s_2)] + 3 - n_2 - n_1 \\ &\leq 2 \log_2 [kd_T(t_1, t_2) + c] + 3 - n_2 - n_1 \\ &\leq 2 \log_2 [(k + c)d_T(t_1, t_2)] + 3 - n_2 - n_1 \\ &\leq 2 \log_2 (k + c) + 2 \log_2 [d_T(t_1, t_2)] + 3 - n_2 - n_1 \\ &\leq \hat{d}_T((t_1, n_1), (t_2, n_2)) + 2 \log_2 (k + c) + 3 \end{aligned}$$

$[(t_1, n_1), (t_2, n_2)]$ *vertical*:

$$\begin{aligned} d_{\hat{S}}((s_1, n_1), (s_2, n_2)) &\leq 2 \log_2 (d_S(s_1, s_2)) - n_2 - n_1 + 3 \\ &\leq 2 \log_2 (k + c) + 2 \log_2 (d_T(t_1, t_2)) - n_2 - n_1 + 3 \\ (1) \quad &\leq 2 \log_2 (k + c) + n_2 - n_1 + 3 \\ &\leq 2 \log_2 (k + c) + \hat{d}_T(t_1, t_2) + 3 \end{aligned}$$

Since $[t_1, t_2]$ is not descending, $\log_2(d_T(t_1, t_2)) \leq n_2$, justifying (1).

The coarse density of the image is clear. Since q is a quasi-isometry, we can also get identical results with a symmetric argument. Thus, \hat{q} is a $(1, 2 \log_2(k + c) + 3)$ -quasi-isometry. \square

Theorem 4.3. Let Γ_1 and Γ_2 be finitely generated groups. Suppose that Γ_1 is hyperbolic relative to a finite collection \mathcal{A}_1 such that that no $A \in \mathcal{A}$ is properly relatively hyperbolic. Let $q : \Gamma_1 \rightarrow \Gamma_2$ be a quasi-isometry of groups. Then there exists \mathcal{A}_2 , a collection of subgroups of Γ_2 , such that the cusped space of $(\Gamma_1, \mathcal{A}_1)$ is quasi-isometric to that of $(\Gamma_2, \mathcal{A}_2)$.

Proof. We extend q to a map $Q : X(\Gamma_1) \rightarrow X(\Gamma_2)$. First, let $Q = q$ on $\text{Cay}(\Gamma_1)$. By the proof of Theorem 5.12 of [Dru09], q induces a quasi-isometric embedding of cosets of elements of \mathcal{A} into those of \mathcal{B} and we can take these to have uniform constants. We observe that this can be made coarsely surjective because no peripheral subgroup is properly relatively hyperbolic.

Still, q might only take a coset to within a bounded distance of the corresponding coset in Γ_2 , rather than directly to it as in Lemma 4.2. We can still use the induced quasi-isometry

on the subsets of the horoballs which have positive depth but at depth 0 we have to make an adjustment.

However, the proof of [Dru09, Theorem 5.12] shows us that there exists a bound T such that the image of q is at most T from the coset to which it is quasi-isometric. Thus, we have to account only for an extra additive T in the constants. Furthermore, T has only the indicated dependencies.

Thus we know that q induces quasi-isometries between cosets of peripheral subgroups and that the constants of these quasi-isometries do not depend on the particular cosets but only on the constants of q and on (Γ_1, \mathcal{A}) . Thus, Q is a quasi-isometry on each individual horoball.

Now let $[x, y]$ be a geodesic arc between points of $X(\Gamma_1)$. We divide this arc into several subarcs by taking the collection of maximal subarcs I_1 which have every vertex of depth 0 and also take the complimentary collection of segments I_2 . In other words, we have

$$[x, y] = [x_0, x_1] \cup [x_1, x_2] \cup \dots \cup [x_{n-1}, x_n] = [x_0, x_n]$$

with $[x_{2i}, x_{2i+1}] \in I_1$ and $[x_{2i+1}, x_{2i+2}] \in I_2$. Essentially, we have divided $[x_0, x_n]$ into segments which go between two different horoballs (possibly with length 0) and segments which traverse individual horoballs. We should mention that the subdivision used here has a parity which suggests that the points x_0, x_n must have depth 0, but that this can easily be remedied by concatenating appropriate segments and performing that same estimate.

We have the following expression:

$$\hat{d}_1(x_0, x_n) = \sum_{i=0}^{n-1} \hat{d}(x_i, x_{i+1}) = \sum_{I \in I_1} \text{length}(I) + \sum_{I \in I_2} \text{length}(I)$$

We construct a path in Γ_2 which tracks the image of $[x_0, x_n]$. For each x_i, x_{i+1} , we take any geodesic in $X(\Gamma_2)$, $[Q(x_i), Q(x_{i+1})) (= [q(x_i), q(x_{i+1}))]$, see Figure 5. Because q is a quasi-isometry between Cayley graphs, $d_1 = \hat{d}_1$ for any segment in I_1 , and $d_2 \geq \hat{d}_2$, we get the following estimate on the lengths of the images of endpoints of segments of I_1 :

$$\begin{aligned} \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) &\leq \sum_{i=0}^{(n-1)/2} d_2(q(x_{2i}), q(x_{2i+1})) \\ &\leq \sum_{i=0}^{(n-1)/2} [kd_1(x_{2i}, x_{2i+1}) + c] = \sum_{i=0}^{(n-1)/2} [k\hat{d}_1(x_{2i}, x_{2i+1}) + c] \end{aligned}$$

By Lemma 4.2, we know that the horoballs paired by q are quasi-isometric and, by [Dru09], that the constants can be chosen uniformly. If we let Λ be the maximum among those constants and k, c , we get the following length estimate:

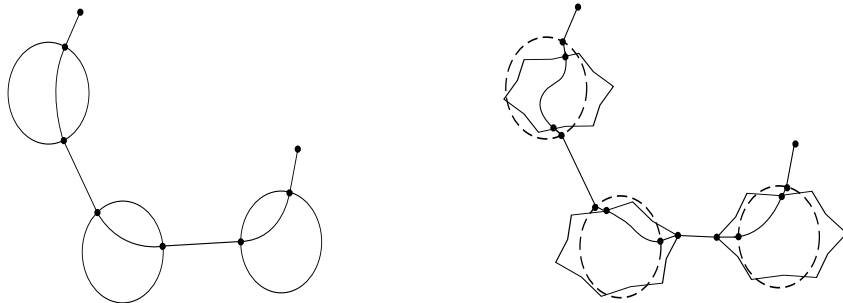


FIGURE 5. A typical geodesic in $X(\Gamma_1)$ and a reconstructed piecewise geodesic in $X(\Gamma_2)$.

$$\begin{aligned}
(2) \quad \hat{d}_2(q(x_0), q(x_n)) &\leq \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) + \sum_{i=1}^{(n-1)/2} \hat{d}_2(q(x_{2i-1}), q(x_{2i})) \\
&\leq \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1}) + \Lambda] + \sum_{i=1}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i-1}, x_{2i}) + \Lambda]
\end{aligned}$$

Now, since we know that the horoball-transversals have length ≥ 1 , we can move the additive constant Λ from the first sum to the second sum, except for a single summand. We do this to account for scenarios in which a geodesic is constructed from several paths contained entirely in horoballs, making the first sum $\sum \Lambda$.

$$\begin{aligned}
\hat{d}_2(q(x_0), q(x_n)) &\leq \Lambda + \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1})] + \sum_{i=1}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i-1}, x_{2i}) + 2\Lambda] \\
&\leq \Lambda + \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1})] + \sum_{i=1}^{(n-1)/2} [3\Lambda \hat{d}_1(x_{2i-1}, x_{2i})] \\
&\leq \Lambda + 3\Lambda \left[\sum_{i=0}^{(n-1)/2} [\hat{d}_1(x_{2i}, x_{2i+1})] + \sum_{i=1}^{(n-1)/2} [\hat{d}_1(x_{2i-1}, x_{2i})] \right] \\
(3) \quad &= 3\Lambda \hat{d}_1(x_0, x_n) + \Lambda
\end{aligned}$$

As in Lemma 4.2, coarse density is clear. By a symmetric argument using the quasi-inverse r ,

$$\hat{d}_1(r(x_0), r(x_n)) \leq 3\Lambda' \hat{d}_2(x_0, x_n) + \Lambda'$$

Therefore,

$$\begin{aligned}
&\hat{d}_1(r(q(x_0)), r(q(x_n))) \leq 3\Lambda' \hat{d}_2(q(x_0), q(x_n)) + \Lambda' \\
&\leq 3\Lambda' [3\Lambda \hat{d}_1(x_0, x_n) + \Lambda] + \Lambda' = 9\Lambda\Lambda' \hat{d}_1(x_0, x_n) + 3\Lambda\Lambda' + \Lambda
\end{aligned}$$

Because q and r are quasi-inverse, there exists a $a > 0$ such that

$$\hat{d}_1(x_0, x_n) \leq \hat{d}_1(r(q(x_0)), r(q(x_n)))$$

Combining these, we get

$$\frac{1}{3\Lambda'} \hat{d}_1(x_0, x_n) - \frac{a + \Lambda'}{3\Lambda'} \leq \hat{d}_2(r(x_0), r(x_n)) \leq 3\Lambda' \hat{d}_1(x_0, x_n) + \Lambda'$$

We conclude by maximizing among constants. \square

As indicated previously, our proof of Theorem 4.3 simplifies to prove an analogous result for the coned space.

Theorem 4.4. *Let $\Gamma_1, \Gamma_2, \mathcal{A}, q$ be as above. Then there exists a collection of subgroups \mathcal{B} of Γ_2 such that the coned spaces of (Γ_1, \mathcal{A}) and (Γ_2, \mathcal{B}) are quasi-isometric.*

Proof. Adjust the proof of the main theorem by replacing the intra-horoball arcs with arcs through the cone-points. These all have length 2 so simply change (2) to

$$\hat{d}_2(q(x_0), q(x_n)) \leq \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) + \sum_{i=1}^{(n-1)/2} 2$$

\square

Corollary 4.5. With $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ as in Theorem 4.3, the cusped spaces $X(\Gamma_1, \mathcal{A}_1)$ and $X(\Gamma_2, \mathcal{A}_2)$ have homeomorphic boundaries.

Corollary 4.6. With $(\Gamma_1, \mathcal{A}_1)$ as in Theorem 4.3, the trees describing the maximal peripheral splitting [Bow98b] and the cut-point/cut-pair tree [PS06] for the boundary of the cusped space are quasi-isometry invariant.

Corollary 4.7. Let S and T be finite generating sets for Γ relative to \mathcal{A} (see [Osi06] for information regarding relative generating sets). Then $X(\Gamma, S, \mathcal{A})$ and $X(\Gamma, T, \mathcal{A})$ are quasi-isometric. The analogous result for the coned space also holds. Furthermore, the hypothesis of Theorem 3.3 that no peripheral subgroup be properly relatively hyperbolic may be discarded.

Proof. Inspecting the proof of Theorem 4.3, the requirement that no peripheral subgroup is properly relatively hyperbolic is used to establish a bijection between cosets of peripheral subgroups from [Dru09]. Because peripheral cosets will be mapped to themselves under the identity map, this bijection is automatic and the hypothesis is unnecessary. \square

We note that this corollary is also new for hyperbolic groups with a non-trivial relatively hyperbolic structure.

5. JSJ DECOMPOSITIONS AND $\partial X(\Gamma)$

With Corollary 4.6 established, the only things remaining to establish of Theorem 5.5 are the identities of the relevant vertex groups. First, we establish the quasi-convexity of vertex groups.

Lemma 5.1. Let Γ be finitely generated and one-ended. Additionally suppose that (Γ, \mathcal{A}) is relatively hyperbolic with \mathcal{A} finite such that no $A \in \mathcal{A}$ is properly relatively hyperbolic and no A contains an infinite torsion subgroup. Let \mathcal{T} be the combined tree from the boundary. If Γ_v is a vertex group of \mathcal{T} then Γ_v is relatively quasi-convex in $X(\Gamma, \mathcal{A})$.

Proof. This is clearly true for vertex groups which are peripheral. It is also true for hyperbolic two-ended vertex groups by Theorem 2.14 and the equivalence of the definitions of quasi-convexity presented in [Hru10] and [MMP10]. Assume Γ_v is not of these types.

\mathcal{T} is a bipartite graph in which all vertices of one color have corresponding groups which are either peripheral or hyperbolic two-ended. Let $\{Z_1, \dots, Z_n\}$ and $\{P_1, \dots, P_m\}$ be the collection of all hyperbolic two-ended subgroups and the collection of all peripheral subgroups incident to Γ_v , respectively.

Let γ be any geodesic between points in Γ_v . Decompose γ into maximal segments of length ≥ 1 which are entirely contained either in cosets of some Z_i , P_j or neither. The segments contained in each Z_i must stay within a bounded distance of Γ_v because hyperbolic two-ended subgroups are strongly relatively quasi-convex (Theorem 2.14). Additionally the P_j segments stay within bounded distance of P_j . Since they are peripheral and we are investigating relative quasi-convexity, we are not concerned with how far they stray from $P_j \cap \Gamma_v$ inside P_j . \square

Proposition 5.2. With $\Gamma, \mathcal{A}, \mathcal{T}$ as in Lemma 5.1, a vertex group Γ_v of \mathcal{T} , is relatively QH with finite fiber if and only if Γ_v is the stabilizer of a necklace in T .

Proof. If Γ_v is relatively QH with finite fiber, then by Definition 2.19 there is a short exact sequence

$$1 \rightarrow F \rightarrow \Gamma_v \rightarrow \pi_1(\mathcal{O}) \rightarrow 1$$

with F finite and \mathcal{O} a hyperbolic orbifold. By Lemma 5.1 it is relatively quasi-convex and by definition it is virtually Fuchsian. Let \mathcal{C} be the set of bi-infinite curves in the universal cover $\tilde{\mathcal{O}}$ which are not homotopic to a boundary component of $\tilde{\mathcal{O}}$. Since F is finite, Γ_v is quasi-isometric with $\pi_1(\mathcal{O})$ and so $\partial\Gamma_v \simeq \partial\pi_1(\mathcal{O})$, call this set N . Let \mathcal{N} be the image of N induced by the inclusion of Γ_v in Γ .

We claim that \mathcal{N} is a necklace in $\partial\Gamma$. By definition, every edge group must be either finite or contained in a boundary component. Because Γ is one-ended, the finite case is excluded. Consequently, for any $\gamma \in \mathcal{C}$ all cosets of edge groups are contained in a single component of $\mathcal{O} \setminus \gamma$. Let $\eta \in \mathcal{C}$ be any curve which has an essential crossing with γ . Such η exists because \mathcal{C} contains no boundary parallel curves. Since η^+ and η^- are in different components of $N \setminus \{\gamma^+, \gamma^-\}$ and each edge group is attached to only a single boundary component, it must

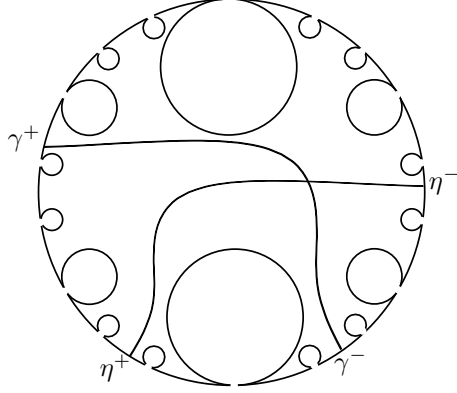


FIGURE 6. Interlocking geodesics separate boundary components.

be that every edge group has image contained in either the same component of $N \setminus \gamma$ as η^+ or η^- , but never both. Thus, the image of $\{\gamma^+, \gamma^-\}$ also separates the image of $\{\eta^+, \eta^-\}$ in \mathcal{N} and the endpoints of γ form a cut pair in $\partial\Gamma$.

In the reverse direction, if Γ_v stabilizes a necklace \mathcal{N} then, by the last paragraph of the proof of [PS06, Theorem 22], it has an action on S^1 which preserves the cyclic order. Since $\Gamma_v = \text{Stab}(\mathcal{N})$, Γ_v inherits the convergence property of Γ and this property must be realized on \mathcal{N} . Any sequence of group elements contained in the kernel of this action is not a convergence sequence, so the kernel must be finite. The fiber, F , is the kernel of this action. By Theorem 2.22, the quotient must be a Fuchsian group. Let \mathcal{O} be the quotient of \mathbb{H}^2 by the action of Γ_v/F , truncating cusps so that \mathcal{O} is compact. We are left with showing that edge groups are boundary parallel.

First, we show that no element of \mathcal{C} (again defined as the set of non-boundary bi-infinite curves on \mathcal{O} - those which cross other bi-infinite curves and interlock in ∂N) can be contained in the image of a peripheral edge group. Because peripheral subgroups have a unique boundary point (Lemma 2.7), such a curve would induce a cut point in \mathcal{N} . However, by Lemma 3.5 no cut pair can be separated by a cut point. In this context, every cut pair separates another cut pair with the only exception arising from those cut pairs which are end points of boundary curves of \mathcal{O} . In other words, for every cut pair C of \mathcal{N} there is an interlocked cut pair unless C forms the endpoints in \mathcal{O} of a curve homotopic to a boundary component of \mathcal{O} .

Now we are left with only the possibility that some $\gamma \in \mathcal{C}$ is identified with a hyperbolic two-ended edge group. Let $\{x_1, x_2\}$ be any cut pair interlocked with $\{\gamma^+, \gamma^-\}$. We claim that $\{x_1, x_2\}$ is not actually a cut pair in $\partial\Gamma$. $\partial\Gamma \setminus \{\gamma^+, \gamma^-\}$ must have at least 3 components in this situation. Let Y be a component which does not contain any points of \mathcal{N} . In particular, $Y \cap \mathcal{N} = \{\gamma^+, \gamma^-\}$. Thus, no cut pair of \mathcal{N} separates γ^+ from γ^- as both are contained in the component Y .

Consider the quotient continuum $Z = \partial\Gamma/\bar{Y}$ with all points of the closure \bar{Y} identified and let x_1 and x_2 be in different components of $\mathcal{N} \setminus \{\gamma^+, \gamma^-\}$. Here, it is easy to see that $\{x_1, x_2\}$ can not be a cut pair in Z for exactly the same reason as in the peripheral case. We now have γ^+ and γ^- as a single cut point in the quotient and we can apply Lemma 3.5. Clearly this still works in $\partial\Gamma$, which demonstrates that Lemma 3.5 can be rephrased as ‘no cut pair is separated by a continuum.’ Thus, we must have that no such γ exists in \mathcal{C} . \square

Lemma 5.3. If $\{x, y\}$ is an inseparable cut pair in $\partial\Gamma$ then $\text{Stab}(\{x, y\})$ is a hyperbolic two-ended subgroup of Γ .

Proof. Let $Z = \text{Stab}(\{x, y\})$. As a subgroup of Γ , Z acts on $\partial\Gamma$ with the convergence property. Since Z fixes $\{x, y\}$, this implies that Z is two-ended. \square

Corollary 5.4. With $\Gamma, \mathcal{A}, \mathcal{T}$ as in Lemma 5.1, there is a correspondence between vertex groups of \mathcal{T} tree and the topological features of the boundary given by

$$\begin{array}{lll} \text{hyperbolic 2-ended} & \longleftrightarrow & \text{cut-pair} \\ \text{peripheral} & \longleftrightarrow & \text{cut-point} \\ \text{relatively QH with finite fiber} & \longleftrightarrow & \text{necklace} \end{array}$$

Proof. Since hyperbolic 2-ended subgroups are strongly relatively quasi-convex [Osi06], their boundaries embed [Hru10]. Similarly, [Bow01] demonstrates that cut points correspond to peripheral splittings. As vertex groups, these features must be separating. The last point is Proposition 5.2. \square

With this in place, we are ready to show:

Theorem 5.5. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose additionally that Γ_1 is one-ended and hyperbolic relative to the finite collection \mathcal{A}_1 of subgroups such that no $A \in \mathcal{A}$ is properly relatively hyperbolic or contains an infinite torsion subgroup. Let \mathcal{T} be the cut-point/cut-pair tree of $\partial(\Gamma_1, \mathcal{A}_1)$. If $f : \Gamma_1 \rightarrow \Gamma_2$ is a quasi-isometry then*

- *T is the cut-point/cut-pair tree for Γ_2 with respect to the peripheral structure induced by Theorem 4.3,*
- *if $\text{Stab}_{\Gamma_1}(v)$ is one of the following types then $\text{Stab}_{\Gamma_2}(v)$ is of the same type,*
 - (1) *hyperbolic 2-ended,*
 - (2) *peripheral,*
 - (3) *relatively QH with finite fiber.*

Proof. By Corollary 4.5, there exists a relatively hyperbolic structure for Γ_2 such that the boundaries of the cusped spaces are homeomorphic. Since T depends only on the topology of this continuum, \mathcal{T} is the cut-point/cut-pair tree for Γ_2 .

By the correspondence given in Corollary 5.4, these vertex types depend only on the topology of the boundary. Since these topological features are preserved, the vertex group types are preserved as well. \square

We conclude with a consequence of the fact that $\text{Out}(\Gamma)$ acts on $\partial\Gamma$ by homeomorphisms.

Corollary 5.6. Let (Γ, \mathcal{A}) be relatively hyperbolic with no $A \in \mathcal{A}$ properly relatively hyperbolic. Then, the $\text{Out}(\Gamma)$ action on the JSJ-deformation space over elementary subgroups relative to peripheral subgroups fixes T .

Proof. The action passes to an action on the boundary by homeomorphisms so that vertex groups map to vertex groups and adjacencies are preserved. Furthermore, because maximal relatively hyperbolic structures are unique [MOY12], there is no change in the choice of peripheral structure. \square

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